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Hilbert's objectivity

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Abstract

Detlefsen (1986) reads Hilbert's program as a sophisticated defense of instrumentalism, but Feferman (1998) has it that Hilbert's program leaves significant ontological questions unanswered. One such question is of the reference of individual number terms. Hilbert's use of admittedly "meaningless" signs for numbers and formulae appears to impair his ability to establish the reference of mathematical terms and the content of mathematical propositions (Weyl, 2009/1949; Kitcher, 1976). The paper traces the history and context of Hilbert's reasoning about signs, which illuminates Hilbert's account of mathematical objectivity, axiomatization, idealization, and consistency.

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Sommario

Detlefsen (1986) legge il programma di Hilbert come una sofisticata difesa dello strumentalismo, ma Feferman (1998) sostiene che il programma di Hilbert lascia senza risposta alcune significative questioni ontologiche. Una fra queste è il riferimento dei termini individuali numerici. L'impiego da parte di Hilbert di simboli per i numeri e formule per la matematica finitista esplicitamente "privi di senso," sembra impedire la possibilità di stabilire un riferimento per i termini matematici e un contenuto per le proposizioni (Weyl, 2009/1949; Kitcher, 1976). Questo articolo ripercorre la storia e il contesto del pensiero di Hilbert concernente i simboli; tale contesto getta luce sulla concezione Hilbertiana dell'oggettività matematica.

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1. The move to finitism

The history of Hilbert's construction of finitist mathematics has a familiar trajectory. Beginning in 1904 and 1905, in response to external pressures from Brouwer's intuitionism and from Poincaré's objections to his consistency argument based on mathematical induction, Hilbert replaces "contentual" mathematics and physics with formal systems (see Mancosu, 1998a, 1998b; Parsons, 1998; Zach, 1998). In response to the

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larger question of justification, Hilbert takes a number of different positions which culminate in the finitist arguments, according to which the reliability of axiom systems, the justified claim that they will not lead us into error, is sufficient to license their use. Reliability is established through proofs of the consistency of the axiom system.

As Weyl (2009/1949) sums it up, in the process, Hilbert's finitism becomes a "meaningless game" of abstract formula manipulation. Weyl did not mean to be derogatory in his reading. Hilbert himself said that the formulae and "number-signs" of finitist mathematics were without meaning. Still, as Weyl and Kitcher (1976) remark, Hilbert's move to finitist methods appears to undermine his account of the reference of mathematical terms, and in consequence of the truth, or at least the objectivity, of mathematical statements.

What is at issue is whether Hilbert's "number-signs" refer in the same way as do terms of ordinary language that refer to objects or entities. Hilbert's contemporaries Gottlob Frege and Aloys Müller base their objections to Hilbert's methods on the requirement that signs should refer to singular objects, as proper names indicate singular persons or entities. According to Benacerraf (1973), we ought to accept this account of mathematical reference because it mirrors ordinary linguistic usage; the truth conditions for mathematical statements ought not be significantly different from the truth conditions for ordinary propositions. Kitcher (1976) and Zach (1998) respond that this criticism appears unfair to Hilbert's own methods.

Detlefsen (1986) mounts a defense of Hilbert's program on the basis that it is a "philosophically sophisticated and convincing defense of mathematical instrumentalism" (p. x). According to Detlefsen, Hilbert's program supports a "new view of how the ontological commitments of mathematics are to be determined" (p. 2); "the ontological commitments of mathematics are located not in those parts of mathematics which we use to acquire knowledge, but rather in those propositions which are used to establish the reliability of the mathematics thus used" (p. 3). Detlefsen's reading of Hilbert's program is, I believe, broadly in agreement with the account presented here. My focus in the paper is on the more specific question of the relationship between the reference of individual mathematical terms and the objectivity of mathematical claims.

While an instrumentalist or reliabilist reading of Hilbert has been well defended, Hilbert's position still can be criticized from a realist perspective. As Feferman (1998) puts it,

Hilbert's idea that mathematical concepts "exist" *only* through axiom systems for them is accepted by very few. For, given that the systems we use are necessarily incomplete (granted their consistency), no such system can be said to fully determine its subject matter. So we are led back to philosophical questions about the nature of mathematical concepts and how we come to accept and have our knowledge about them, questions that are just the sort that Hilbert hoped to avoid by his consistency and completeness programs.

[pp. 14–15]

While Feferman's point is well taken, he is evaluating whether Hilbert's program can answer questions of contemporary interest. My aim is to describe some of Hilbert's own epistemological concerns, influences, and goals. The effort to do so is rewarded by casting light on why Hilbert himself might have thought that an instrumentalist position on mathematical ontology is defensible; and why it may be defensible now, as a position with limited application.

In the tradition with which Hilbert was engaged, evaluating the truth conditions for ordinary perceptual reports and for the basic statements of physics involved constructing statements using signs. The definition of 'sign' with which Hilbert was most familiar is from the sign and depiction (picture) theories of Hermann von Helmholtz and Heinrich Hertz, respectively. In this tradition, a sign that does not copy properties of an external object can be employed to express thoughts with content within a given framework. Hilbert revises

Hertz's methodology under the rubric of the "existential axiomatics" that Hilbert developed with Bernays, which assumes the existence of the referents of a set of signs for the purposes of constructing a proof.

Reading Hilbert's move to finitism in this context illuminates his use of "number-signs" in existential axiomatics more generally. Hilbert makes a similar, and well-known, move in his foundations of geometry (see [Blanchette, 1996](#); [Demopoulos, 1994](#); [Hallett, 2010](#); [Resnik, 1973/1974](#); [Toepell, 1986](#)). In his correspondence with Frege, he argues that the reference of the terms "point," "line," and "plane" is not determined by ostensive definition, but rather by implicit definition. Euclid defines a point as "that which has no parts," but Hilbert's methodology allows for implicit definition via the axioms, which set the rule that a point is the intersection of two lines. Further, within Hilbert's framework, the terms "point," "line," and "plane" can be instantiated by almost anything – a point can have parts. Frege suggested using his pocket watch as a point to demonstrate the problems with Hilbert's approach; Hilbert responded that Frege's suggestion showed how flexible and strong Hilbert's foundations of geometry were (also see [Hilbert, 2004/1894](#)).

Hilbert's "existential axiomatics" covers both the foundations of geometry and the finitist foundations of mathematics, including the foundations of calculation with the integers. Hilbert's finitist methods and his methods in the foundations of geometry are both axiomatic methods of problem-solving using signs. These methods allow for a characteristic way of understanding the reference of terms in a theory, whether mathematical or physical. The paper concludes with a comparison of Hilbert's axiomatic methods in physics and in mathematics. In both cases, the objectivity of proven claims rests, not on the reference of individual terms, but on the reliability of proofs within an axiomatic system.

2. Finitist mathematics and the problem of signs

Hilbert begins his 1922 lectures on the "New Grounding of Mathematics" with a discussion of the motivation for constructing finitist mathematics. He argues that Weyl and Brouwer "dismember and mutilate" mathematics by prohibiting irrational numbers, functions, Cantorian higher number classes, and others ([Hilbert, 1996/1922, 1119](#)). These remarks recall Hilbert's well-known "Mathematical Problems" lecture of 1900, where he said that an "important nerve" of mathematics would be "extirpated" by removing the problems posed by physical theory, including problems that hinge on the continuum or on irrational numbers ([Hilbert, 1996/1900, 1098ff](#)). Ordinary, "concrete-intuitive" mathematics can proceed by operations on sets of objects. But the problems Hilbert is interested in lead beyond the powers of concrete mathematics. Finitist methods are intended to take over where concrete-intuitive mathematics leaves off: finitist mathematics holds out the promise of dealing rigorously with the infinite, the continuous, the irrational.

In concrete-intuitive mathematics, the numbers can be considered objects. But an infinite series is not concrete and surveyable, so it is not possible to give an explicit account of infinite series as sets of objects. Nonetheless, for Hilbert, finitist mathematics, like concrete mathematics, must begin with a set of operations on "surveyable" objects.

[A]s a precondition for the application of logical inferences and for the activation of logical operations, something must already be given in representation: certain extra-logical discrete objects, which exist intuitively as immediate experience before all thought. If logical inference is to be certain, then these objects must be capable of being completely surveyed in all their parts, and their presentation, their difference, their succession (like the objects themselves) must exist for us immediately, intuitively, as something that cannot be reduced to something else.

[[Hilbert, 1996/1922, 1121](#)]

Unlike the concrete objects of intuitive number theory, the "objects" of finitist mathematics are signs. Hilbert says that the sign 1, the sign +, and strings of these (1 + 1, 1 + 1 + 1, and so on) are the numbers, the "objects," of finitist mathematics. He continues,

Because I take this standpoint, the objects of number theory are to me—in direct contrast to Frege and Dedekind—the signs themselves, whose shape can be generally and certainly recognized by us—independently of space and time, of the special conditions of the production of the sign, and of insignificant differences in the finished product. The solid philosophical attitude that I think is required for the grounding of pure mathematics—as well as for all scientific thought, understanding, and communication—is this: *In the beginning was the sign.*¹

Hilbert's refusal to give an account of the reference, in the Fregean sense, of the concepts of number and of proof drew immediate criticism. It is at the heart of the first two objections from Kitcher and Weyl cited above, but it was raised much earlier by the philosopher Aloys Müller. Müller's criticism, and Hilbert's and Bernays's responses, set the stage for the interpretation of Hilbert in the 20th century. In 1923, Müller wrote a response to Hilbert's "New Grounding", an essay called "On Numbers as Signs". The essay begins with a quotation from Hilbert's own remarks in the "New Grounding":

The sign 1 is a number.

A sign, that begins and ends with 1, such that in between + always follows 1 and 1 always follows +, is likewise a number, for instance, the signs

$$1 + 1$$

$$1 + 1 + 1$$

These number-signs, which are numbers and make up the numbers completely, are themselves the object of our consideration, but otherwise have no *meaning*. In addition to these signs we employ other signs, which *mean* something and serve as communication, for instance the sign 2 as a short form of the number-sign $1 + 1$, or the sign 3 as a short form of the number-sign $1 + 1 + 1$; further, we use the signs =, >, which serve to communicate an assertion. So $2 + 3 = 3 + 2$ is not a formula, but only serves as a communication of the fact that $2 + 3$ and $3 + 2$, with respect to the short forms employed, are the same number-sign, namely the number-sign $1 + 1 + 1 + 1 + 1$. No more is $3 > 2$ a formula, rather, it serves only as a communication of the fact that the sign 3, that is, $1 + 1 + 1$, projects beyond the sign 2, that is, $1 + 1$, or that the latter sign is a part of the former.²

Hilbert's "number-signs" are employed within an axiomatic framework, to generate strings of number-signs. These strings are then designated by other signs. "2" and "3" stand in for " $1 + 1$ " and " $1 + 1 + 1$ ", respectively.

Aloys Müller criticizes Hilbert's statement that the number-signs "1" and "+" have "no meaning":

A sign always *describes* something, *which is distinct from the sign itself*. Sign and designated object are *coordinated* with each other. If one speaks of a "sign without meaning", then the coordinated object is missing, and the word "sign" necessarily has another sense. [...] Thus, if Herr Hilbert wishes to maintain that 1 and + are without meaning, then they are not signs, rather, in these cases we have to do merely with indications, figures, or, as we would rather put it, shapes.³

In Müller's view, a sign must designate some object, with which it is "coordinated", if it is to be a sign at all. Müller (1923) makes several related objections:

¹ Hilbert (1996/1922, 1121–1122), emphasis in original.

² Hilbert (1996/1922, 1122), translation amended.

³ Müller (1923, 154), translated for this essay.

1. “First, it is characteristic of Herr Hilbert’s account that he declines to found number theory on set theory and takes numbers as irreducible objects” (p. 153).
2. A “sign” is always coordinated with another object. The phrase “meaningless sign” is itself meaningless (p. 154).
3. The fact that we are familiar with the numerals, and already associate a meaning with them, allows Hilbert to use them as “signs” without an adequate account. If we used $^{*}x^{*}$, or $xoxoxo$, to denote $1 + 1 + 1 + 1$, then we would see that the finitist strategy of using “meaningless” “number-signs” is flawed (pp. 154–155).
4. To say “ $3 > 2$ ” is true because $1 + 1 + 1$ “projects beyond” $1 + 1$, Müller argues, is not justified, because Hilbert does not give an adequate definition of “projects beyond” (p. 156).
5. Thus, “*mere shapes do not suffice as a basis for number theory*” (p. 157).
6. “Now if 1 is a real number, what does that mean? That an object must be given, that is distinct from it, but is coordinated with it, as the object’s sign. [...] What kind of objects do the signs 1, 2, 3 describe? That is the fundamental question of the theory of mathematical objects.”⁴

Müller’s objections reveal what to him was a real tension in Hilbert’s statements about finitism. The tension is between Hilbert’s insistence that signs are “concrete” and “intuitable,” objects of “immediate” intuition in Kant’s sense, and his statement that signs have “no meaning.”

In this context, one might recall Hilbert’s remark that for him, as opposed to for Frege and for Dedekind, the objects of number theory are “the signs themselves.” Müller’s objection boils down to the following. If signs are concrete, intuitable objects and are “without meaning,” then they must function as signs *in virtue of* their concrete, intuitable properties, that is, in virtue of their observable form or shape. But if those shapes don’t indicate other objects, they no longer function as signs at all, only as brute shapes or figures. But Hilbert says the signs *are* the numbers. In that case, the foundation of number theory on Hilbert’s view is the manipulation of shapes that do not indicate any further object or phenomenon: “mere” “signs without meaning.”

Müller argues that this feature of Hilbert’s account is a serious limitation of finitist methods. A general objection along Müller’s lines can be put informally as follows. First, Hilbert claims that the signs are identical with their concrete, directly observable properties (their shape). But Hilbert also claims that the signs are identical to the numbers and “make them up completely.” Therefore, Hilbert’s account of how signs function as numbers must appeal only to their concrete, directly observable properties (their shape). However, this general argument, which is behind Müller’s objections, is not valid. For it to be valid, one would need to add a premise stating that “Hilbert’s account of how signs function as numbers is limited to Hilbert’s account of the signs themselves.”

In the axiomatic tradition in which Hilbert was working, however, this assumed premise is not necessarily true. The section following (Section 3) will examine Müller’s ideal-realist theory of signs and their meaning, and will draw a contrast between that theory and the Helmholtz–Hertz sign theory, and the broader axiomatic tradition, in which Hilbert was working. The final section of the paper (Section 4) will explain how Hilbert’s finitist methods were adopted to solve certain problems inspired by natural science.

3. Ideal-realism and the sign theory

The debate over the reference of terms in Hilbert’s theory turns on Hilbert’s admission that the terms he used for the integers in finitism were “signs” without meaning. Clearly, then, much turns on what Hilbert meant by the word “sign.” Hilbert’s distinctive approach is illuminated by taking a closer look at the history

⁴ Müller (1923, 157), all translations for this essay.

of nineteenth century views of signs and of their role in axiomatized theories. Hilbert's account of the reference of mathematical terms was informed by the sign theory [*Zeichentheorie*] of Hermann Lotze, Hermann von Helmholtz, and others; and by the theory that succeeded it, the theory of depiction [*Bildtheorie*] of Heinrich Hertz.

Aloys Müller's view, on the other hand, is known as ideal-realism [*Idealrealismus*]. Müller (1913) describes ideal-realism as a blend of idealism and "strict" realism (p. 1). Wundt (1904) describes the view as follows:

Insofar as the direction of this philosophy takes as its starting point to grasp the entire real world as a necessary development of the progressive logical consistency of thought, the system can be described overall with a word that Fichte used for his own system: ideal-realism. Above all, this expression makes explicit the attribute that is the characteristic distinguishing feature of this new form of idealism as opposed to all previous ones from Plato to Kant: the complete rejection of the opposition between being and appearance.⁵

For an ideal-realist, being and appearance are complementary elements of perception and of thought. The domain of real things cannot be isolated from the domain of perception, as the things that are perceived just are the real things. The sign theory, in contrast, is based upon the Kantian view that things as they appear to us are not things as they are in themselves.⁶ This epistemological difference between ideal-realism and the sign theory turns out to be significant for the theory of signs and their role in axiomatized theories.

Hermann von Helmholtz developed a naturalist account of the epistemology of sensing over his long career in physiology, physics, mathematics, and chemistry. Helmholtz's views on sense-perception and on perceptual reports, and its precursor the *Zeichentheorie* of Hermann Lotze and Johannes Müller, was a preeminent account of observation-statements when Hilbert was first working.⁷ Johannes Müller argued for a "specific sense energy," according to which the properties of a sense organ determine the quality and intensity of the resulting sensation, not exclusively the properties of the external stimulus. Lotze argued for a theory of "local signs," which incorporated the notion of specific sense energy, but added the view that each sensation can be "localized" in a larger space of sensation and experience (Lotze, 1852, 330ff). Localization is achieved by using sensations as "signs" of the external world. For instance, a sensation of heat in a certain place is a sign of the presence of fire near that place. Lotze points out that the sensations are not the things they represent: the feeling of heat is not the fire itself, and a sensation of light does not illuminate anything. However, one can use the associations between sign and signified, which are learned and not innate, to navigate the objects and phenomena revealed in experience.

Helmholtz incorporates Lotze's view of local signs in his empirist account of the physiology of perception. Helmholtz argues that the brain makes "unconscious inferences" from sign to thing signified, which allow us to navigate, but that these inferences are learned, and not innate. Helmholtz takes sensations to be signs of external objects, and claims that "our sensations are qualitatively only *signs* of the external objects, and certainly not *copies* with any degree of similarity."⁸ In the first edition of his *Handbook on Physiological Optics*, Helmholtz discusses the epistemological significance of his theory of perception.⁹ Here, Helmholtz repeats this radical statement about signs: "I have described sense perceptions only as

⁵ P. 399, translation for this essay.

⁶ Helmholtz's mature view was empiricist (or empirist), as Hatfield (1990) observes, and so not generally compatible with transcendental idealism.

⁷ See Baird et al. (1998), Boring (1942), Lenoir (1993), and Hatfield (1990), among others, for accounts of the history of the sign theory.

⁸ Helmholtz (1968/1869, 56).

⁹ Helmholtz (1867, 442ff). I am grateful to Gary Hatfield for mentioning §26 as relevant here.

symbols for the relationships of the external world, and denied them any kind of similarity or equality with that which they describe” (Helmholtz, 1867, 442).

But, Helmholtz observes, this does not answer the larger question of whether our representations agree with external phenomena more generally, as a group or system. Helmholtz is opposed to nativism, according to which there is a “pre-established harmony” between nature and our mind, and sensualism, according to which representations are entirely deceptive. Instead, he urges, “our intuitions and representations are effects, which the intuited and represented objects have brought about on our system of nerves and our consciousness” (Helmholtz, 1867). Representations depend on the nature of our senses (Müller’s specific sense energies), but they do not entirely deceive us. However, Helmholtz denies to our sensations any particular *epistemic* access to external reality; rather, he argues, there is only a *practical* agreement between sensations and reality (Helmholtz, 1867, 443).

This practical agreement is similar to Lotze’s view that we use sensations as local signs to navigate the external world. For Helmholtz, “representation and represented belong to two entirely distinct worlds” (Helmholtz, 1867). The only connection between the two worlds is a set of causal relationships, which guide our action. In the second edition of the *Handbook*, Helmholtz argues that we can have access to the lawful ordering principles of the phenomena through reasoning about signs, even though propositions that use signs may not correspond to reality:

I need not explain to you that it is a *contradictio in adjecto* to want to represent the real [...] in positive terms but without grasping it in our forms of representation. This is often discussed. What we can achieve is knowledge of the lawful order in the realm of the actual, this, certainly, only presented in the sign system of our sense impressions.

[Helmholtz, 1896, 593]

For Helmholtz, we are not presented with a set of immediate perceptions of the independent properties of substances (Helmholtz, 1896, 591). We navigate the external world by discovering “the *lawful* in the phenomena,” that is, a correspondence between the sequences of our perceptions and the sequence of occurrences in nature. One can represent this lawful regularity or correspondence by means of signs, and then use the signs to construct representations of the phenomena that capture the lawful relationships (Helmholtz, 1896, 594). Again, these laws are not epistemically justified – they are justified only insofar as they can function as practical guides to the external world.

In the *Principles of Mechanics*, published in 1894, Helmholtz’s student Heinrich Hertz extends the sign theory in the context of his theory of depiction (*Bildtheorie*), with a particular focus on capturing lawful regularities in mechanics. A depiction is a formal representation of the observed phenomena, derived deductively from a set of postulates: a basic principle or principles, axioms, and definitions.

[W]hen we speak simply and generally of the principles of mechanics [...] by this will be meant any selection from amongst such and similar propositions, which satisfies the requirement that the whole of mechanics can be developed from it by purely deductive reasoning without any further appeal to experience.¹⁰

A principle can be acquired through inductive reasoning. Hertz uses the principle of inertia in his mechanics, which is based partly on empirical evidence. But when that principle is used to construct a depiction of the observed phenomena, all the results that can be derived from that principle, the axioms, and the definitions are, or should be, “purely deductive” consequences of the principle.

The first aim of a system governed by a principle is to recapture the theorems of mechanics as deductive consequences of the principle, axioms, and definitions. The final goal of such a system is to represent the

¹⁰ Hertz (1956/1894, 4–5).

relations between phenomena to argue that necessary relations between elements of the system likewise are necessary in nature.

We make ourselves internal apparent likenesses or symbols of external objects, and indeed we make them of such a kind, that the necessary sequences in thought of the depictions always are depictions of the necessary sequences in nature among the objects depicted. So that this requirement should be met generally, certain conformities must exist between nature and our minds. Experience teaches us that this requirement is satisfiable, and, therefore, that such conformities in fact obtain.¹¹

Hertz's depictions are constructed so that, if successful, the sequences of depictions of the phenomena mirror the sequences of the phenomena. When a theory succeeds in giving such a depiction, Hertz argues, that is the justification for claiming that the theorems proved within the theory are objective. That justification does not depend on the reference of the terms of the axiom system, but rather, on the "conformities" "between nature and our minds," that is, between regularities in the phenomena of interest and statements describing those regularities in axiomatic theories.

Hertz's proofs of the "conformity" between nature and our minds may seem reminiscent of the nativists' pre-established harmony. But Hertz's proofs are indirect, not direct. They are performed by demonstrating features of the axiom system used to construct a depiction. For instance, Hertz distinguishes between the "correctness" of a depiction of the phenomena, its accuracy in describing what was observed, and the "fitness to the purpose" of that depiction, how well it is fit to the purpose of capturing the relations between the target phenomena.

Two [logically] permissible and correct depictions of the same external objects may yet differ in respect of fitness to the purpose. Of two depictions of the same object that is the more fit which depicts more of the essential relations of the object,—the one which we may call the more distinct. Of two depictions of equal distinctness the more fit is the one which contains, in addition to the essential characteristics, the smaller number of superfluous or empty relations,—the simpler of the two. Empty relations cannot be avoided entirely.¹²

The two components of fitness to the purpose are distinctness and simplicity. Given two depictions, one that has the smaller number of "superfluous and empty relations" is simpler. One that depicts more of the "essential relations" of the object under investigation is more distinct. The depiction remains a depiction, though: even a maximally simple and distinct system will contain *some* empty relations, necessary only to the depiction.

A depiction containing idealizations can be more fit to the purpose of capturing essential relations simply than a depiction containing only direct observational reports. Theories should account for all the relevant observed phenomena. But two equally empirically adequate theories might employ different axioms and definitions to represent those phenomena. An axiomatization that uses idealization to simplify the explanation (account for the phenomena using fewer tools) is superior, in Hertz's system, to an axiomatization that does not use idealizations but that uses more conceptual relations to account for the phenomena. Moreover, a system that uses idealizations as instruments may be able to capture more of the "essential relations" of interest, and in this case as well, an axiom system using idealization is justified.

The sign-theoretic approach is inconsistent with Aloys Müller's ideal-realist epistemology of number theory. Recall Wundt's point, that ideal-realism rejects the Kantian distinction between appearances and

¹¹ Hertz (1956/1894, 1), translation amended.

¹² Hertz (1956/1894, 2), translation amended.

things in themselves. Müller goes further in his own writings on ideal-realism; while his view initially sounds Kantian, it appeals to a commonality between “depiction and original.”

We can call phenomenal actuality a depiction of the transcendent system of reality. Here the concept of depiction is taken quite generally. A is a *depiction* of B means, that a certain *coordination* exists from A to B (in our case, it exists on the basis of the mediated or unmediated emergence of A with the help of B). A depiction is always a synthesis, that is, the characteristics of the original that is depicted, and the reality, on [the basis of] which depiction takes place, merge in the depiction. *From the determinate concept of coordination in our case it follows that depiction and original have something in common.* This that is in common we describe as *invariant*; the word is chosen because one can regard the depiction as a transformation of the original.¹³

The italicized claim differentiates Müller’s view from Helmholtz’s. They agree that depictions are a synthesis of external object and means of representation. But Müller argues that in some cases we can conclude that “depiction and original have something in common,” which is precisely what Helmholtz denies.

The key difference between Müller’s ideal-realism and Helmholtz’s and Hertz’s sign and depiction theories, then, is the relationship between sign and depicted object. For Müller, there must be a relationship of “coordination” between sign and object for the sign to function as part of a depiction of objective reality, where “coordination” requires something sign and object have in common. For Helmholtz, we cannot know that perceptual signs and their objects have any particular feature in common, since the representation of the sign depends on our sensory apparatus and its interaction with the object, as well as on independent properties of the object. For Hertz, we can know that there are “conformities” between nature and our representations of it, but those conformities are found, not in relationships between signs and objects, but in the laws of mechanics and of physics.

Müller’s epistemological position on the coordination between signs and objects cuts off finitist mathematics at the root. Finitism was constructed to solve problems that cannot be solved in “concrete-intuitive” mathematics, including problems having to do with infinite series, irrational numbers, and complex numbers. It is not possible to display a “coordination” between an infinite series and its object, for instance, because neither an infinite series, nor an object of infinite extension, is concretely observable *in toto*. It is possible that Müller, like Kronecker, wishes to deny that the infinite, complex, and irrational belong in number theory. While Müller was not as explicit about this as was Kronecker, it does appear to be a consequence of his position on “coordination” of sign with object.

4. Problem-solving and access to mathematical knowledge

4.1. *The motivation for finitism: mathematical problems*

Hilbert’s motivation for finitism explicitly was to be able to deal with problems, including problems involving infinite series and irrational numbers, that go beyond observation. These problems are often inspired by physics and by natural science more generally. In his lecture “Mathematical Problems” delivered in Paris in 1900, Hilbert begins with a programmatic statement of the relationship between “external phenomena” and mathematical reasoning. He argues that, while mathematics springs from experience, mathematicians are able to go beyond the problems suggested by experience, to evolve new problems independently, and to investigate the phenomena more deeply.

¹³ Müller (1913, p. 2); translated for this essay, emphasis added.

Surely the first and oldest problems in every branch of mathematics spring from experience and are suggested by the world of external phenomena [...] But, in the further development of a branch of mathematics, the human mind [...] evolves from itself alone, often without appreciable influence from without, by means of logical combination, generalization, specialization, by separating and collecting ideas in fortunate ways, new and fruitful problems, and appears then itself as the real questioner.¹⁴

Hilbert objects to the view, which he attributes to Kronecker, that only arithmetic and analysis, but not calculations with irrational or infinite numbers, can be rigorous. As Hilbert observes,

Such a one-sided interpretation of the requirement of rigor would soon lead to the ignoring of all concepts arising from geometry, mechanics, and physics. [...] But what an important nerve, vital to mathematical science, would be cut by the extirpation of geometry and mathematical physics! On the contrary I think that wherever, from the side of the theory of knowledge or in geometry, or from the theories of natural or physical science, mathematical ideas come up, the problem arises for mathematical science to investigate the principles underlying these ideas and so to establish them upon a simple and complete system of axioms, that the exactness of the new ideas and their applicability to deduction shall be in no respect inferior to those of the old arithmetical concepts.¹⁵

Here, Hilbert uses Hertz's terminology for "fitness to the purpose": an axiom system that is "simple" and "complete" is one that accounts for all the known phenomena using a maximally small number of relations.¹⁶

As Weyl points out, Hilbert thinks that geometry and physics can be brought into connection as axiomatized theories; and, further, that every scientific domain can be axiomatized.

Geometry and physics may be adjoined, as soon and insofar as they have been strictly axiomatized. Hilbert even believes (*Axiomatisches Denken*, 1917), "Every potential subject of scientific thought, as soon as it is ripe for the formation of a theory, is bound to fall under the axiomatic method and, therefore, indirectly to the lot of mathematics."

[Weyl, 2009/1949, 60]

According to Hilbert, physics derives its certainty from mathematics. That certainty does not come about from the particular conclusions of mathematics, but from axiomatic approaches to mathematical problem solving. Weyl continues,

In the same sense as Hilbert, [...] Husserl (*Logische Untersuchungen*, I, §71) declares with particular reference to mathematical logic that "the mathematical form of treatment [...] is for all strictly developed theories [...] the only scientific one, the only one that affords systematic completeness and perfection and gives insight into all possible questions and their possible forms of solution.

[Weyl, 2009/1949, 60n]

For Hilbert, access to mathematical objectivity is relative to the mathematical problems that can be solved using the axiomatic method, as will be discussed in the section following.

¹⁴ Hilbert (1996/1900, 1098).

¹⁵ Hilbert (1996/1900, 1098ff).

¹⁶ Sieg (2009): "Hilbert's view of the geometric axioms as characterizing a system of things that presents a 'complete and simple image of geometric reality' is, after all, complemented by a traditional one: the axioms must allow to establish, purely logically, all geometric facts and laws" (p. 338).

4.2. *Existential axiomatics and idealization*

Hilbert presented his views about the relationship between contentual mathematics, “existential axiomatics,” and idealization a number of times in the 1920s. And, of course, [Bernays and Hilbert \(1934\)](#) exemplifies Hilbert’s axiomatic methodology.¹⁷ First, and perhaps surprisingly, for Hilbert a part of elementary mathematics is contentual. The number-signs are concrete and surveyable, and we can make statements about them that communicate content, in the sense that our thoughts and proofs are *about* the signs themselves.¹⁸ This is the meaning of “contentual” for Hilbert: that the specific object of one’s assertions can be produced and surveyed. All that is asserted about the numbers in elementary mathematics is that they are concrete objects with a certain shape. In fact, Bernays agreed with Müller that “figure” is a better word than “sign” ([Bernays, 1996/1923](#)). As Hilbert could in his debate with Frege, Bernays could give up ground without giving up the axiomatic methodology: Hilbert admitted to Frege that a pocket watch could be a point, and Bernays admitted to Müller that figure is a better word than sign for the integers of Hilbert’s elementary mathematics. Hilbert’s geometrical axioms are supposed to lay bare the geometrical relationships involved in the foundations of geometry. Hilbert’s elementary mathematics is supposed to make explicit the relationships between the integers, and to allow for “contentual” statements about them. The integers need only be concrete, observable, singular objects – they need not refer to anything beyond their own figure or shape.

As Hilbert points out, though, this move works only for the integers. Even basic algebra requires calculations with formulas and variables that do not have a unique observable content. Some elements of a formula may refer to a determinate content, while others will not:

Hence even elementary mathematics contains, first, formulas to which correspond contentual communications of finitary propositions (mainly numerical equations or inequalities, or more complex communications composed of these) and which we may call the *real propositions* of the theory, and, second, formulas that—*just like the numerals of contentual number theory*—in themselves mean nothing but are merely things that are governed by our rules and must be regarded as the *ideal objects* of the theory.

[470, emphasis added]

The number-signs of elementary number theory and the formulas of algebra ($1 + a = a + 1$) have in common that they mean nothing independently, but are manipulable within the axiomatic system, and are “governed by our rules.”

Hilbert’s methods of manipulating signs and formulas “without meaning” are connected intimately to his axiomatic method, which he developed independently and in cooperation with Bernays. In the “New Grounding,” Hilbert describes the essence of the axiomatic method as: “In order to investigate a subfield of a science, one bases it on the smallest possible number of principles, which are to be as simple, intuitive, and comprehensive as possible, and which one collects together and sets up as axioms” ([Hilbert, 1996/1922](#), 1119). The terminology “simple, intuitive, and comprehensive” is reminiscent of Hertz, as well as of noted axiomatizers of mechanics Gustav Kirchoff and Ludwig Boltzmann ([Corry, 1997](#), 92ff; [Jungnickel and McCormach, 1990](#), 125ff). The term “simple” has an epistemic, not aesthetic significance for Hertz and the others. First, simplicity is always comparative: one axiom system is simpler than another if the two are both empirically adequate but one uses fewer internal relationships (e.g. fewer axioms, definitions, laws) to depict the target phenomena. Second, that an axiom system is simpler than another is an epistemic indication, that the simpler system is closer to depicting the real relationships than the more complex one.

¹⁷ [Pasch \(1882\)](#) is a significant influence on Hilbert’s axiomatic methods; see also [Corry \(1997\)](#).

¹⁸ See, e.g., [Hilbert \(1967/1927, 467–469\)](#) and [Hilbert \(1996/1922, 1119–1121\)](#).

Hilbert and Bernays add to this tradition an analysis of what it means to formulate the axioms and to construct an axiom system from them. In the *Foundations of Mathematics*, they observe that

a refinement that the axiomatic standpoint has received in Hilbert's *Foundations of Geometry* consists in the following: that in the axiomatic construction of a theory, from the factual materials of representation on which the basic concepts of a theory are developed one retains only extracts which are formulated in the axioms, and abstracts from all other content.

[Bernays and Hilbert, 1934, 1, trans. for this essay]

There is a choice of which content to represent in the axioms. The choice is significant, for it brackets “all other content” on which the theory is based.

In axiomatics in the narrowest meaning, the *existential form* comes along as a further moment. Through this *the axiomatic method* is distinguished from the *constructive* or *genetic* methods of founding a theory. While with the constructive method objects of the theory are inserted only as a *class* of things, in an axiomatic theory one has to do with a fixed system of things (or, as the case may be, several such systems) which make up a *domain of subjects, limited* from the outset, for all predicates from which the statements of the theory are assembled. In the presupposition of such a totality of the “individual-domain” – disregarding the trivial cases, in which a theory only has to do with a finite, strictly limited collection of things anyway – is found an idealizing assumption, which is found in the assumptions formulated through the axioms.

[Bernays and Hilbert, 1934]

An axiomatization collects the “smallest number” of principles possible, and assumes only some elements of the “factual,” “material” content of the theory to be axiomatized. Limiting the “domain of subjects” with which the axiom system will deal has the result of fixing a “system of things” that makes up a “totality.”

Instead of “generating” or “constructing” definitions or classes of things to be dealt with by the theory, Hilbert's and Bernays's “existential axiomatics” instead makes an “idealizing assumption” at the outset: “the presupposition of a totality of the ‘individual - domain’.” An axiomatization based on such an assumption thus is capable of proving that the axiom system is consistent *given* the assumption of a certain fixed content. Moreover, Hilbert and Bernays endeavor to show that the real propositions or proofs of a given theory can go through even when the ideal elements of the theory are removed from the specific proofs. This strategy resembles Gödel's notion of “outer consistency.”¹⁹

Hilbert's use of proofs of outer consistency, of idealizing assumptions, and of the method of ideal elements illuminates the response to Müller's objections hinted at above. Müller's view depends on the following argument, stated informally. Hilbert claims that the signs are identical with their concrete, directly observable properties (their figure or shape). But Hilbert also claims that the signs are identical to the numbers and “make them up completely.” Therefore, Hilbert's account of how signs function as numbers must appeal only to their concrete, directly observable properties (their shape). Again, for this objection to be valid, one would need to add a premise stating that “Hilbert's account of how signs function as numbers is limited to Hilbert's account of the signs themselves.”

But which signs? There are the “1”s and “+”s of the initial presentation of the “New Grounding.” These are the “concrete,” “meaningless” signs to which Müller objects. But Hilbert also appeals to logical signs, to algebraic signs, and to others; in fact, it is a deep methodological commitment of finitism that signs stand in, not only for content, but also for idealizing assumptions of various kinds. In “Mathematical Problems,”

¹⁹ Gödel himself thought establishing outer consistency was “central” to Hilbert's programs (Feferman's, Solovay's and Webb's note to Gödel, 1990/1972, 286). An anonymous reviewer suggested that I emphasize Hilbert's and Bernays's employment of idealizations, and of the notion of outer consistency. I am grateful for this suggestion, which greatly improved the account of axiomatization in the paper.

Hilbert argues that the use of signs to designate concepts or phenomena is necessary to analysis and to geometry as well as to physics.

To new concepts correspond, necessarily, new signs. These we choose in such a way that they remind us of the phenomena which were the occasion for the formation of the new concepts. So the geometrical figures are signs or mnemonic symbols of space intuition and are used as such by all mathematicians. Who does not always use along with the double inequality $a > b > c$ the picture of three points following one another on a straight line as the geometrical picture of the idea ‘between’? [. . .] Or who would give up the representation of the vector field, or the picture of a family of curves or surfaces with its envelope which plays so important a part in differential geometry, in the theory of differential equations, in the foundation of the calculus of variations and in other purely mathematical sciences? The arithmetical symbols are written diagrams and the geometrical figures are graphic formulae; and no mathematician could spare these graphic formulae, any more than in calculation the insertion and removal of parentheses or the use of other analytical signs.

[Hilbert, 1996/1900, 1098–1099]

While Hilbert argues here that the use of signs is “indispensable,” that does not mean that he thinks any proof depends on the form of a particular sign. In the Helmholtz–Hertz tradition, signs are used to depict the lawlike relationships between the phenomena. In Hilbert’s methodology, signs might be used as the integers, or to denote an idealizing assumption, a logical operator, an algebraic variable, or a formula. In some cases, signs are “contentual,” even in the trivial sense that they refer to themselves. In other cases, signs denote ideal elements of a theory, which are indispensable to calculation in mathematics and physics.

Hilbert begins with the assumption of what he and Bernays call existential axiomatics, namely, the assumption that a set of objects exist on which operations can be performed, and then proceeds to assign signs to stand in for these objects, and to construct proofs using those signs (Bernays and Hilbert, 1934, 1ff). As Bernays puts it, for Hilbert,

A constructive reinterpretation of the axioms of existence is not possible just in the way that one converts them into generative principles for the construction of numbers, rather, the manner of proof made possible through such an axiom can be replaced entirely by a formal process such that certain signs take the place of general concepts such as number, function, and so forth. Where concepts are missing, in due time a sign will emerge. This is the methodological principle of the Hilbertian theory.

[Bernays, 1922, 16]

Sieg (2009) remarks, “The word finitist is intended to convey the idea that a consideration, a claim or definition respects (i) that *objects are representable*, in principle, and (ii) that *processes are executable*, in principle” (p. 361). The “in principle” is significant. In finitist mathematics, a “number-sign” designates a mathematical object that could be represented. If numbers were not representable, and generatable, in principle using the axiomatic system, then the proofs of finitist mathematics would be impossible. The rigor of a proof using signs for ideal elements depends on proofs that the choice of sign does not affect the proof, i.e., on proofs of outer consistency.

5. Conclusion

This paper began with a comparison of Detlefsen’s (1986) defense of Hilbert as a mathematical instrumentalist and Feferman’s (1998) criticism of Hilbert’s program from a contemporary ontological perspective. Feferman had argued that “no such [axiomatic] system can be said to fully determine its subject matter [given incompleteness]. So we are led back to philosophical questions about the nature of mathematical concepts.” However, if we consider Hilbert’s axiomatic program, not as an attempt to give a thoroughgoing determination of the subject matter of mathematics, but as a program of giving a foundation for reliable

methods of problem solving and of proofs of objectivity, then Feferman's criticism, while still valid, loses some of its bite.

As Feferman remarks, Hilbert was interested in giving consistency and completeness proofs in order to lay to rest questions about the infinite, for instance, which cannot be solved by appeal to "concrete" mathematics. Hilbert himself thought that the reference of terms in an axiomatic system can be established by means of implicit definition. Feferman's criticism is that the reference of the terms cannot be determined completely, given the incompleteness of any system beyond first-order arithmetic.

But if one reads Hilbert as an instrumentalist, as Detlefsen and others do, this is not a fatal blow to Hilbert's program. For Hilbert's goal was not to describe the properties of the numbers, or to determine completely the subject matter of number theory. Instead, to paraphrase Weyl, it was to give insight into the character of certain mathematical and physical problems, and of their reliable means of solution. This can be done within the axiomatic system, even if the terms used in that system are "meaningless," conventional signs. On the adoption of the instrumentalist reading, there are certainly still questions remaining about the "nature of mathematical concepts," but Hilbert's program can give a persuasive account of the reliability of mathematical methods.

Finally, the "ideal" part of finitist mathematics does not make up Hilbert's entire program. As [Detlefsen \(1986\)](#) describes the distinction between real and ideal propositions,

For Hilbert, the apparent propositions and proofs of mathematics are to be divided into two groups: (a) those whose epistemic value derives from the evidentness of their content (the so-called *real* or *contentual* propositions and proofs), and (b) those whose epistemic value derives from the role that they play in some formal algebraic, or calculary scheme (the so-called *ideal* or *non-contentual* pseudo-propositions and pseudo-proofs).

[p. 4]

As Detlefsen describes it, Hilbert's position on this score leads to the well-known disagreement with Poincaré over metamathematics, and in particular over the status of the principle of induction. The use of signs for ideal elements in finitist mathematics belongs to "non-contentual" mathematics.

It is not the case that we may take *no* epistemic attitude toward the propositions of finitist mathematics that use signs for real or ideal elements, however. The "epistemic value" of such theorems, or inferences more generally, is that they illuminate the character of the problems to be solved, in mathematics and in physics. They do not (necessarily) allow us to make particular judgments about the numbers, or to describe the properties of the numbers. But they may point the way to solutions to problems. While solving particular problems may not yield all the knowledge we seek, about the character of mathematical concepts and objects, or about the reference to reality of mathematical statements, it does give us knowledge that we ought to value instrumentally, as a step on the path to developing reliable mathematical methods.²⁰

²⁰ An anonymous reviewer for this journal made concise, invaluable suggestions for revision and for incorporating new material (especially the notion of "outer consistency"), which have had a profound effect on the final version. This paper first took shape in a seminar Wilfried Sieg allowed me to audit as a visiting faculty member in 2009, and would have been neither conceived nor completed were it not for Professor Sieg's generous assistance and guidance, and the stimulating atmosphere in the seminar. Much earlier versions of the paper were presented at the Eighth Conference of HOPOS, and at &HPS3, at which there was much productive discussion and I received many useful suggestions from Michael Friedman, Gary Hatfield, Jeremy Heis, Don Howard, Graciela de Pierris, Bryan Roberts, Michael Stoeltzner, Sean Walsh, and others. All errors of fact and of reason in the paper remain my own.

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